NORMS WITH LOCALLY LIPSCHITZIAN DERIVATIVES

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ABSTRACT
If a separable Banach space X admits a real valued function ϕ with bounded nonempty support, ϕ' is locally Lipschitzian and if no subspace of X is isomorphic to c_0, then X admits an equivalent twice Gateaux differentiable norm whose first Frechet differential is Lipschitzian on the unit sphere of X.

1. Introduction

In this paper we study some stronger smoothness properties of real Banach spaces. Throughout the paper X will be a real Banach space and, usually, we consider spaces which have no subspace isomorphic to c_0. (Subspaces are always closed.)

We begin with some notation and definitions. A function $\phi : X \to \mathbf{R}$, the reals, with bounded nonempty support is called a bump function. If there is such a function ϕ on X with ϕ' locally uniformly continuous on X (resp., locally Lipschitz on X) then X is said to be *locally uniformly smooth* (LUS) (resp., *locally Lipschitz smooth* (LLS)). Correspondingly, we say X is *uniformly smooth* (US) (resp., *Lipschitz smooth* (LS)) if the above properties are global. Above and throughout the paper, unless otherwise stated, all derivatives are taken in the Fréchet sense.

A Banach space X is of *type 2* (see [10]) if there is $C < \infty$ such that for every finite set $x_1, x_2, \cdots, x_n$,

$$\sum_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \leqq 2^n C \sum_{i=1}^{n} \| x_i \|^2 .$$

X is *super-reflexive* [11] if only reflexive spaces Y are finitely representable in X, that is, for each $\varepsilon > 0$ and finite dimensional such space $F \subset Y$ there is an

[†] This author's research supported in part by NSERC (Canada) Grant A7535.
Received February 11, 1982 and in revised form August 22, 1982

isomorphism T of F onto $T(F) \subseteq X$ such that $\|T\| \, \|T^{-1}\| \leqq 1+\varepsilon$. By the Enflo–James theorem [6], X is super-reflexive if and only if X admits an equivalent *uniformly rotund norm*, that is, $\lim \|x_n - y_n\| = 0$ whenever $x_n, y_n \in X$, $\{x_n\}$ bounded and

$$\lim [2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2] = 0.$$

Further, the norm $\|\cdot\|$ is *locally uniformly rotund* (LUR) provided $\lim \|x - x_n\| = 0$ whenever $x_n, x \in X$ and $\lim [2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2] = 0$. Kadec (see [5]) has shown that every separable Banach space admits an equivalent LUR norm. (Since we are concerned only about equivalent norms, usually we will omit the word equivalent.) $x \in K$, $K \subseteq X$ and convex, is a *strongly exposed point* [16] of K if there is $f \in X^*$ such that $f(x) = \sup\{f(y) : y \in K\}$ and if $x_n \in K$, $\lim f(x_n) = f(x)$, then $\lim \|x_n - x\| = 0$.

2. Preliminary results

2.1 Proposition. *If X is a Banach space satisfying*:

(i) *X admits an* LUR *norm*;

(ii) *X is* LUS (resp., LLS);

(iii) *No subspace of X is isomorphic to c_0.*

Then X is US (resp., LS).

In the proof we will use the following two lemmas.

2.2 Lemma. *If $\|\cdot\|$ is an* LUR *norm on X and $K \subseteq X$ is compact, then for each $\varepsilon > 0$ there is $\delta > 0$ such that for every finite $F \subseteq K$ there is a finite codimensional subspace $H \subseteq X$ such that for each $h \in H$, $\|h\| \geqq \varepsilon$, we have $\|x + h\| > \|x\| + \delta$ for each $x \in F$.*

Proof. Let $x_n \in K$ and $h_n \in f_n^{-1}(0)$ where f_n supports $B(0, \|x_n\|)$ at x_n, i.e., $\|f_n\| = 1$ and $f_n(x_n) = \|x_n\|$. First we will show that if $\|x_n + h_n\|^2 - \|x_n\|^2 \to 0$, then $h_n \to 0$. We may assume $x_n \to x_0 \neq 0$. Then

$$\begin{aligned} 0 &\leqq 2\|x_0\|^2 + 2\|x_n + h_n\|^2 - \|x_0 + x_n + h_n\|^2 \\ &\leqq 2\|x_0\|^2 + 2\|x_n\|^2 - |f_n(x_0 + x_n + h_n)|^2 + 2\|x_n + h_n\|^2 - 2\|x_n\|^2 \\ &\leqq 2\|x_0\|^2 + 2\|x_n\|^2 - |2\|x_n\| - \|x_n - x_0\||^2 + 2\|x_n + h_n\|^2 - 2\|x_n\|^2. \end{aligned}$$

So, $\lim(2\|x_0\|^2 + 2\|x_n + h_n\|^2 - \|x_0 + x_n + h_n\|^2) = 0$. Thus, by LUR, $\|x_n + h_n - x_0\| \to 0$ and, consequently, $h_n \to 0$.

Therefore, we have that for every $\varepsilon > 0$ there is $\delta > 0$ such that $x \in K$,

$h \in f_x^{-1}(0)$ where f_x supports $B(0,\|x\|)$ at x, $\|h\| \geqq \varepsilon$ implies $|\|x+h\|-\|x\|| \geqq \delta$. Further, since $\|x+h\| \geqq f_x(x+h) = f_x(x) = \|x\|$, $\|x+h\| \geqq \|x\|+\delta$. The proof can now be finished easily.

The following lemma is similar to lemma 2.2 in [21].

2.3 LEMMA. *Let $\|\cdot\|$ be an* LUR *norm on X and let $\phi : X \to \mathbf{R}$ be such that $\phi(0)=0$ and ϕ' is locally uniformly continuous on X. Assume X is not* US *and set $\mathcal{F} = \{F \subseteq X$: card $F < \infty$ and $\Phi(x) \leqq \|x\|$ for all $x \in F\}$. For $F \in \mathcal{F}$, let*

$$Q(F) = \{h \in X : \|h\| \leqq \tfrac{1}{2}, \phi(x \pm h) \leqq \|x \pm h\| \text{ for each } x \in F\}.$$

Assume $K \subseteq X$ is compact. Then there exists a $\delta_K > 0$ such that for every $F \subseteq K$ such that $F \in \mathcal{F}$ there exists $h \in Q(F)$ such that $\|h\| \geqq \delta_K$.

PROOF. For each $x \in K$ there is $\varepsilon_x > 0$ so that ϕ' is uniformly continuous on $B(x,\varepsilon_x)$. Then $K \subseteq \bigcup_{i=1}^m B(x_i, \varepsilon_{x_i}/2)$, $x_i \in K$. Let $\varepsilon = \min\{\varepsilon_{x_i} : i = 1,\cdots,m\}$. For $x \in K$, ϕ' is uniformly continuous on $B(x,\varepsilon/2) \subseteq B(x_i,\varepsilon_{x_i})$, for some i. Let $F = \{y_1,\cdots,y_n\} \subseteq K$, $F \in \mathcal{F}$, and consider the function

$$\psi(h) = \sum_{i=1}^{n} ([\phi(y_i+h)-\phi(y_i)]^2 + [\phi(y_i-h)-\phi(y_i)]^2).$$

Then ψ' is uniformly continuous on $\|h\| < \varepsilon/2$. By Lemma 2.2, for K, F and $\varepsilon > 0$ as given, there exist $0 < \delta < \varepsilon$ and a finite codimensional subspace $H \subseteq X$ so that $\|x+h\| > \|x\| + \delta$ for each $x \in F$ and each $h \in H$, $\|h\| > \varepsilon$.

Suppose that for each $h \in H$, $\varepsilon/4 < \|h\| < \varepsilon/2$, we have $\psi(h) > \delta^2$. Then if τ is a C^∞-function on $\mathbf{R}$ such that $\tau(t) = 0$ for $|t| \geqq \delta^2$, $\tau(0) = 1$, then $\tau \circ \psi$ has a uniformly continuous differential on $B(0,\varepsilon/2)$, $\tau(\psi(0)) = 1$ and $\tau(\psi(x)) = 0$ for $x \in H$, $\varepsilon/4 < \|x\| < \varepsilon/2$. Setting $\tau \circ \psi \equiv 0$ for $\|x\| \geqq \varepsilon/2$ we obtain a function on H with bounded support and uniformly continuous differential. Since $X = H \oplus H_1$, $\dim H_1 < \infty$, we can construct such a bump function on X which contradicts the assumption that X is not US. Thus, there is $h_0 \in H$, $\varepsilon/4 < \|h_0\| < \varepsilon/2$ such that $\psi(h_0) \leqq \delta^2$. Then

$$\phi(y_i \pm h_0) \leqq \phi(y_i) + \delta \leqq \|y_i\| + \delta \leqq \|y_i \pm h\| \quad \text{for each } y_i \in F.$$

So $h_0 \in Q(F)$ and $\delta_K > 0$.

We note that a similar result can be proved for Lipschitz smoothness.

PROOF OF PROPOSITION. (We will prove it for the US case; the other case is similar.) Let $\|\cdot\|$ be an LUR norm on X no subspace of which is isomorphic to c_0, and let $\phi : X \to \mathbf{R}$, $\phi(0) = 0$, $\phi(x) = 3$ for $\|x\| \geqq 1$ and ϕ' is locally uniformly continuous. Assume X is not US; we will obtain a contradiction using a technique introduced in [4] and [13] and used in [21].

A sequence $\{h_n\} \subseteq X$ is constructed inductively such that $\phi(\Sigma_{i=0}^{n} \varepsilon_i h_i) \leqq \|\Sigma_{i=0}^{n} \varepsilon_i h_i\|$ for each $n = 0, 1, \cdots$ and each choice of $\varepsilon_i = \pm 1$, as follows: $h_0 = 0$ and if $h_0, \cdots, h_n$ have been constructed put $F_n = \{\Sigma_{i=0}^{n} \varepsilon_i h_i : \varepsilon_i = \pm 1\}$. Choose $h_{n+1} \in Q(F_n)$, by Lemma 2.3, so that $h_{n+1} > \frac{1}{2}\sup\{\|h\| : h \in Q(F_n)\}$.

Now, if $\|\Sigma_{i=0}^{n} \varepsilon_i h_i\| \leqq 1$ for each n and each choice of $\varepsilon_i = \pm 1$, we obtain a contradiction. For, since no subspace of X is isomorphic to c_0, by a result of Bessaga and Pelczynski ([1], cf. [17], p. 98), Σh_i is unconditionally convergent. Hence the partial sum $\{\Sigma_{i=1}^{n} \varepsilon_i h_i : n = 0, 1, \cdots, \varepsilon_i = \pm 1\}$ is relatively compact. So, by Lemma 2.3,

$$\inf_n \|h_n\| \geqq \tfrac{1}{2} \inf_n \sup\{\|h\| : h \in Q(F_n)\} > 0$$

which contradicts the convergence of Σh_i.

On the other hand, suppose that for some n and some choice ε_i, $\|\Sigma_{i=0}^{n} \varepsilon_i h_i\| > 1$. Let n be the first integer such that $\|\Sigma_{i=0}^{n} \bar{\varepsilon}_i h_i\| > 1$ for some choice $\bar{\varepsilon}_i$. Then $\|\Sigma_{i=0}^{n-1} \bar{\varepsilon}_i h_i\| \leqq 1$ and, since $\|h_i\| < \frac{1}{2}$, $\|\Sigma_{i=0}^{n} \bar{\varepsilon}_i h_i\| < 1 + \frac{1}{2} < 3$. Thus, $\phi(\Sigma_{i=0}^{n} \bar{\varepsilon}_i h_i) \leqq \|\Sigma_{i=0}^{n} \bar{\varepsilon}_i h_i\| < 3$ which contradicts that $\phi(x) = 3$ for $\|x\| \geqq 1$.

The following lemma presents a Lipschitz variant of the Banach–Smulian characterization of smooth points on the unit sphere of a Banach space.

2.4 LEMMA. *For $y \in X$ let f_y denote a support functional of $B(0, \|y\|)$ at y. For each $x \in X$, $\|x\| = 1$, the following are equivalent*:

(i) *There is $C > 0$ such that for all $h \in X$,*

$$\|x + h\| - \|x\| - f_x(h) \leqq C\|h\|^2.$$

(ii) *There is $C > 0$ such that for $h \in X$,*

$$\|x + h\| + \|x - h\| - 2\|x\| \leqq C\|h\|^2.$$

(iii) *There is $C > 0$ such that for $g \in X^*$, $\|g\| \leqq 1$,*

$$\|g - f_x\| \leqq C(1 - g(x))^{1/2}.$$

(iv) *There is $C > 0$ such that for $y \in B(0,1)$,*

$$\|f_x - f_y\| \leqq C\|x - y\|.$$

(v) *There is $C > 0$ such that for $y \in X$ and $g \in \delta(\frac{1}{2}\|\cdot\|^2)(y)$, $\|g - f_x\| \leqq C\|x - y\|$, where $\delta(\frac{1}{2}\|\cdot\|^2)(y)$ denotes the subdifferential of $\frac{1}{2}\|\cdot\|^2$ at y.*

(vi) *There is $C > 0$ such that for all $h \in X$,*

$$\|x+h\|^2+\|x-h\|^2-2\|x\|^2<C\|h\|^2.$$

PROOF. The equivalence of (i) and (ii) is standard. To show that (i) implies (iii), let $g\in X^*$. If $h\in X$, $\|h\|=1$ and $t>0$, then $t(g-f_x)(h)=g(x+th)-\|x\|-f_x(th)+1-g(x)<\|x+th\|-\|x\|-f_x(th)+1-g(x)< Ct^2+1-g(x)$, by (i). Hence,

$$\begin{aligned}\|g-f_x\|&\leqq \inf\{Ct+t^{-1}(1-g(x)):t>0\}\\&=C^{1/2}(1-g(x))^{1/2}.\end{aligned}$$

(iii) implies (iv), for, if $y\in B(0,1)$, then, by (iii),

$$\begin{aligned}\|f_y-f_x\|&\leqq C(1-f_y(x))^{1/2}\leqq C(1-f_y(x)+\|y\|-f_x(y))^{1/2}\\&=C((f_y-f_x)(y-x))^{1/2}\leqq C\|f_y-f_x\|^{1/2}\cdot\|y-x\|^{1/2}.\end{aligned}$$

If $y=0$, then (v) follows immediately from (iv). If $y\neq 0$ and $g\in\delta(\frac{1}{2}\|\cdot\|^2)(y)$, then $g\|y\|^{-1}\in\delta(\|\cdot\|)(y\|y\|^{-1})$. Thus, assuming (iv),

$$\begin{aligned}\|g\|y\|^{-1}-f_x\|&\leqq C\|y\|y\|^{-1}-x\|<C(\|y-x\|+\|y\|y\|^{-1}-y\|)\\&=C(\|y-x\|+|\,\|y\|-1|)\leqq 2C\|y-x\|.\end{aligned}$$

Further, $\|g-g\|y\|^{-1}\|=\|g\|\cdot\|y\|^{-1}(|\,\|y\|-1|)=|\,\|y\|-1|\leqq\|y-x\|$. Thus, $\|g-f_x\|\leqq\|g-g\|y\|^{-1}\|+\|g\|y\|^{-1}-f_x\|\leqq(2C+1)\|y-x\|$, and (v) obtains.

To show that (v) implies (vi), fix $h\in X$. By the mean value theorem, there is $0\leqq t\leqq 1$ and $g\in\delta(\frac{1}{2}\|\cdot\|^2)(x+th)$ such that $\|x+h\|^2-\|x\|^2=2g(h)$. By (v), $\|x+h\|^2-\|x\|^2-2f_x(h)=2(g-f_x)(h)\leqq 2\|g-f_x\|\cdot\|h\|\leqq 2C\|h\|^2$. The same inequality holds for $-h$ and adding the inequalities we get (vi).

Finally, (vi) implies (ii):

$$\begin{aligned}\|x+h\|+\|x-h\|&\leqq(2\|x+h\|^2+2\|x-h\|^2)^{1/2}\\&\leqq(4\|x\|^2+4C\|h\|^2)^{1/2}\leqq 2\|x\|+2C\|h\|^2.\end{aligned}$$

2.5 DEFINITION. (a) $f:X\to\mathbf{R}$ is said to be *twice Gateaux differentiable* at $x\in X$ if

(i) $f'(y)h$, the first Gateaux differential, exists, is linear and continuous in h for y in some neighbourhood of x;

(ii) $f''(x)(h,k)=\lim_{t\to 0}(1/t)[f'(x+tk)h-f'(x)h]$ exists for each h, $k\in X$ and is a continuous, bilinear, symmetric form in h and k.

(b) $x\in S_1=\{x\in X:\|x\|=1\}$ is called a *Lipschitz smooth point* of S_1, if any one of the conditions in Lemma 2.4 is satisfied.

The following theorem is implicitly contained in [24], [23] and [21]. For sake of completeness and because it represents a basic technique of proving a smoothness characterization of spaces isomorphic to Hilbert space, we present a proof (based on one in [21]) of this result. We will need the following lemma from [21, lemma 2.2].

2.6 LEMMA. *Let H be a finite codimensional subspace of X. Then there is a finite codimensional subspace $G \subseteq X^*$ such that for each $g \in G$ there is $x \in H$, $x \neq 0$, such that $g(x) > \frac{1}{3}\|g\| \cdot \|x\|$.*

2.7 THEOREM. *Assume that there is a Lipschitz smooth point $x_0 \in S_1$. Further, assume that the dual norm on X^* is twice Gateaux differentiable on the unit sphere in X^*. Then X is isomorphic to a Hilbert space.*

PROOF. Suppose that the assumptions obtain and that X is not isomorphic to a Hilbert space. Let $\phi(x) = \|x\|$ on X and $\psi(x^*) = \|x^*\|$ on X^*. For $x_0^* \in X^*$ such that $x_0^*(x_0) = 1$, $\|x_0^*\| = 1$, define $H = (\phi'(x_0))^{-1}(0) \cap (x_0^*)^{-1}(0) \subseteq X$. By Lemma 2.6, there is a finite codimensional subspace $G_1 \subseteq X^*$ such that for any $g \in G_1$ there is a nonzero $x \in H$ such that $g(x) > \frac{1}{3}\|g\| \cdot \|x\|$. Let $G = G_1 \cap \psi'(x_0^*)^{-1}(0) \cap x_0^{-1}(0) \subseteq X^*$. Since codim $G < \infty$, G cannot be isomorphic to a Hilbert space, by our assumptions on X.

Thus, $\inf\{|\psi''(x_0^*)(g,g)| : g \in G, \|g\| = 1\} = 0$, for otherwise, $\psi''(x_0^*)(g,g)$ would yield an equivalent inner product norm on X, a contradiction. Since x_0 is a Lipschitz smooth point of S_1, there is $C > 0$ such that for each $t \in \mathbf{R}$, $h \in X$,

$$\phi(x_0 + th) - \phi(x_0) - \phi'(x_0)(th) < Ct^2\|h\|^2. \tag{1}$$

Choose $g \in G$, $g \neq 0$, such that $\psi''(x_0^*)(g,g) < (1/100C)\|g\|^2$. By the Taylor formula,

$$\psi(x_0^* + tg) = \psi(x_0^*) + \psi'(x_0^*)(tg) + \tfrac{1}{2}\psi''(x_0^*)(tg, tg) + \sigma(t), \tag{2}$$

where $\lim_{t\to 0}[\sigma(t)/t^2] = 0$ and the derivatives are taken in the Gateaux sense. Choose $h \in H$ with $\|h\| = (1/10C)\|g\|$ and

$$g(h) \geqq \frac{1}{3}\|g\| \cdot \|h\| = \frac{1}{30C}\|g\|^2.$$

By (1) and (2), we obtain

$$\|x_0 + th\| < 1 + \frac{t^2\|g\|^2}{100C} \quad \text{and} \quad \|x_0^* + tg\| < 1 + \frac{1}{2} \cdot \frac{t^2\|g\|^2}{100C} + \sigma(t).$$

Thus, we have

$$1 + t^2\|g\|^2\frac{1}{30C} \leqq 1 + t^2 g(h) = (x_0^* + tg)(x_0 + th)$$

$$\leqq \|x_0^* + tg\| \cdot \|x_0 + th\| \leqq \left(1 + \frac{t^2\|g\|^2}{100C}\right)\left(1 + \frac{t^2\|g\|^2}{200C} + \sigma(t)\right)$$

$$= 1 + \frac{3}{200C}\, t^2\|g\|^2 + \sigma_1(t)$$

where $\lim_{t\to 0}[\sigma_1(t)/t^2] = 0$. This is a contradiction and the theorem is proved.

The next theorem is a variant of Lindenstrauss' result ([16]) on the density of smooth points on the sphere of certain spaces.

2.8 THEOREM. (Lindenstrauss) *If $(X,\|\cdot\|)$ is reflexive and admits a norm $|\cdot|$ for which all of the points of its unit sphere are Lipschitz smooth points, then the set of all Lipschitz smooth points of the original unit sphere S_1 is dense in S_1.*

PROOF. Using Lemma 2.4 above, follow the proof of theorem 3 in [16].

2.9 COROLLARY. *$l_p(\mathbf{N})$, $1 < p < 2$, does not admit an equivalent twice Gateaux differentiable norm.*

PROOF. Let q satisfy $1/p + 1/q = 1$. The usual norm on $l_q(\mathbf{N})$ has modulus of smoothness $\rho(\tau) \leqq C\tau^2$, so, by Lemma 2.4, all the points of the unit sphere in $l_q(\mathbf{N})$ are Lipschitz smooth. Now assume there is an equivalent twice Gateaux differentiable norm on $l_p(\mathbf{N})$. By Theorem 2.8, the corresponding dual unit sphere has Lipschitz smooth points and, by Theorem 2.7, $l_p(\mathbf{N})$ is isomorphic to Hilbert space which is a contradiction.

3. Main results

3.1 THEOREM. *Assume that X is a separable Banach space which admits a differentiable norm whose differential is locally Lipschitz on the unit sphere S_1. Then X admits a twice Gateaux differentiable norm whose first differential is locally Lipschitz on S_1.*

PROOF. Let $\phi_0 : \mathbf{R} \to \mathbf{R}$ be a C^∞-function, $\phi_0 \geqq 0$, even, $\phi_0 \equiv 0$ outside $[-\frac{1}{2}, \frac{1}{2}]$ and $\int_{\mathbf{R}} \phi_0 = 1$. Let $\phi_n(t) = 2^n \phi_0(2^n t)$ for $t \in \mathbf{R}$ and let $\{h_i\}$, $i = 0,1,2,\cdots$, be a dense sequence in $S_1 \subseteq X$. Now define a sequence $\{f_n\}$ of functions on X by:

$$f_0(x) = \|x\|^2,$$
$$f_{n+1}(x) = \int_{\mathbf{R}^{n+1}} f_0\left(x - \sum_{i=0}^{n} t_i h_i\right) \cdot \phi_0(t_0) \cdot \phi_1(t_1) \cdots \phi_n(t_n)\, dt_0\, dt_1 \cdots dt_n$$
for $n = 0, 1, 2, \cdots$.

This sequence of functions has the following properties:

(i) Each f_n is a convex and continuous function. This is an immediate consequence of the facts that f_0 is convex and $\phi_n \geqq 0$.

(ii) There is a function $f: X \to \mathbf{R}$ such that $f_n \to f$ uniformly on bounded sets, f is Frechet differentiable with a locally Lipschitz differential and

$$f'(x)(h) = \lim \int_{\mathbf{R}^{n+1}} f_0'\left(x - \sum_{i=0}^{n} t_i h_i\right)(h) \cdot \prod_{i=0}^{n} \phi_i(t_i) dt_0 \cdots dt_n \quad \text{for } x, h \in X.$$

To see this, let $m \geqq n$ and consider $\|x\| \leqq r$. Then,

$$\begin{aligned}
&|f_m(x) - f_n(x)| \\
&\quad = \left\| \int_{\mathbf{R}^{m+1}} \left[f_0\left(x - \sum_{i=0}^{m} t_i h_i\right) - f_0\left(x - \sum_{i=0}^{n} t_i h_i\right)\right] \prod_{i=0}^{m} \phi_i(t_i) dt_0 dt_1 \cdots dt_m \right\| \\
&\quad \leqq \int_{|t_i| \leqq 1/2^{i+1}} \left\| \sum_{n+1}^{m} t_i h_i \right\| \left(2\|x\| + \left\| \sum_{i=0}^{m} t_i h_i \right\| + \left\| \sum_{i=0}^{n} t_i h_i \right\|\right) \prod_{i=0}^{m} \phi_i(t_i) dt_0 \cdots dt_m \\
&\quad \leqq (2r+2)/2^n.
\end{aligned}$$

Thus, by the Cauchy criterion, $f_n \to f$ uniformly on $B(0, r)$, for some $f: X \to \mathbf{R}$.

Now, $f_m'(x)h = \int_{\mathbf{R}^{m+1}} f_0'(x - \Sigma_{i=0}^{m} t_i h_i)(h) \Pi_{i=0}^{m} \phi_i(t_i) dt_0 \cdots dt_m$ for each x, $h \in X$. Since $\bigcup_{k=1}^{\infty} \{\Sigma_{i=0}^{k} t_i h_i : |t_i| \leqq 1/2^{i+1}\}$ is relatively compact and f_0' is locally Lipschitzian, it can be seen that for each $x_0 \in X$ there is $\delta_{x_0} > 0$ such that f_0' is Lipschitzian on $B(x_0, \delta_{x_0}) - \bigcup_{k=1}^{\infty} \{\Sigma_{i=0}^{k} t_i h_i : |t_i| \leqq 1/2^{i+1}\}$, with Lipschitz constant C_{x_0}. Therefore, as above, if x, $y \in B(x_0, \delta_{x_0})$,

$$\begin{aligned}
|(f_m'(x) - f_m'(y))h| &\leqq \int_{\mathbf{R}^{m+1}} C_{x_0} \|x - y\| \prod_{i=0}^{m} \phi_i(t_i) dt_0 \cdots dt_m \\
&= C_{x_0} \|x - y\|.
\end{aligned}$$

Finally, if $x \in B(x_0, \delta_{x_0})$, $m \geqq n$ and $\|h\| \leqq 1$, then

$$\|(f_m'(x) - f_n'(x))h\| < \int_{\mathbf{R}^{m+1}} C_{x_0} \left\| \sum_{i=n+1}^{m} t_i h_i \right\| \prod_{i=0}^{m} \phi(t_i) dt_0 \cdots dt_m \to 0$$

uniformly on $B(x_0, \delta_{x_0})$ as n, $m \to \infty$. So $f'(x)h$ exists in the Fréchet sense and is locally Lipschitzian in X. Moreover,

$$f'(x)h = \lim_n \int_{\mathbf{R}^{n+1}} f_0'\Big(x - \sum_{i=0}^n t_i h_i\Big)(h) \prod_{i=0}^n \phi_i(t_i)\,dt_0 \cdots dt_n.$$

(iii) f is a twice Gateaux differentiable function on X.

Let $0 \leqq i \leqq n$ be fixed, h, $x \in X$ and $m \geqq n$. Then

$$\lim_{\alpha \to 0} \int_{\mathbf{R}^{m+1}} \frac{1}{\alpha}\Big[f_0'\Big(x + \alpha h_i - \sum_{i=0}^m t_i h_i\Big)h - f_0'\Big(x - \sum_{i=0}^m t_i h_i\Big)h\Big] \prod_{i=0}^m \phi_i(t_i)\,dt_0 \cdots dt_m$$

$$= \lim_{\alpha \to 0} \int_{\mathbf{R}^{m+1}} \frac{1}{\alpha}\Big[f_0'(x - t_0 h_0 - \cdots - (t_i - \alpha)h_i - \cdots - t_m h_m)h$$

$$- f_0'\Big(x - \sum_{i=0}^m t_i h_i\Big)h\Big] \cdot \prod_{i=0}^m \phi_i(t_i)\,dt_0 \cdots dt_m$$

$$= \lim_{\alpha \to 0} \int_{\mathbf{R}^{m+1}} f_0'\Big(x - \sum_{i=0}^m t_i h_i\Big)(h)\phi_0(t_0) \cdots \frac{(\phi_i(t_i + \alpha) - \phi_i(t_i))}{\alpha} \cdot$$

$$\cdot \phi_{i+1}(t_{i+1}) \cdots \phi_m(t_m)\,dt_0 \cdots dt_m$$

$$= \int_{\mathbf{R}^{m+1}} f_0'\Big(x - \sum_{i=0}^m t_i h_i\Big)(h) \cdot \phi_0(t_0) \cdots \phi_i'(t_i) \cdots \phi_m(t_m)\,dt_0 \cdots dt_m.$$

Using an argument similar to that in (ii) above, we see that

$$\lim_m \int_{\mathbf{R}^{m+1}} f_0'\Big(x - \sum_{i=0}^m t_i h_i\Big)(h)\phi_0(t_0) \cdots \phi_i'(t_i) \cdots \phi_m(t_m)\,dt_0 \cdots dt_m$$

exists uniformly for $x \in B(x_0, \delta_{x_0})$, $\|h\| \leqq 1$, for fixed i. Thus, for each $x \in X$, $i \in \mathbf{N}$ and $h \in X$,

$$f''(x)(h, h_i) = \lim_n \int_{\mathbf{R}^{n+1}} f_0'\Big(x - \sum_{i=0}^n t_i h_i\Big) h\phi_0(t_0) \cdots \phi_i'(t_i) \cdots \phi_n(t_n) \cdot dt_0 \cdots dt_n.$$

Further, since the Fréchet differential f' is locally Lipschitzian and $\{h_i\}$ is a dense set of directions, $f''(x)(h,k) = \lim_i f''(x)(h,k_i)$ uniformly on $\|h\| \leqq 1$ and $x \in B(x_0, \delta_{x_0})$ whenever $k_i \to k$. Also, since $f''(x)(h,k_i)$ is locally Lipschitzian in x for each i, $f''(x)(h,k)$ is, for h, k fixed, continuous in x. Thus, $f''(x)(h,k)$ is, for a fixed x, a bounded, symmetric, bilinear form in h, k.

(iv) Let q be the Minkowski functional of the set $Q = \{x \in X : f(x) \leqq 16\}$. We claim that q is an equivalent twice Gateaux differentiable norm on X.

This follows, since if $\|x\| > 5$, then $f(x) > 16$. Also $f(0) < 16$ and f is a continuous, convex function. Thus, q is an equivalent norm on X. Further, if $\|x\| > 5$, then $f'(x)(x) > 12$ and $f'(-x)(-x) > 12$, since, setting $a = \sum_{i=0}^n t_i h_i$, we have $\|a\| < 1$ and, for $0 < \alpha < \frac{1}{2}$,

$$\frac{1}{\alpha}(\|(1+\alpha)x-a\|^2-\|x-a\|^2)$$

$$=\frac{1}{\alpha}(\|(1+\alpha)x-a\|+\|x-a\|)(\|(1+\alpha)x-a\|-\|x-a\|)$$

$$\geqq\frac{4}{\alpha}(\|(1+\alpha)x-a\|-\|x-a\|)=\frac{4}{\alpha}(\|(1+\alpha)(x-a)+\alpha a\|-\|x-a\|)$$

$$\geqq\frac{4}{\alpha}((1+\alpha)\|x-a\|-\alpha\|a\|-\|x-a\|)=4(\|x-a\|-\|a\|)\geqq 12.$$

Hence

$$f'(x)(x)=\lim_n\int_{\mathbf{R}^{n+1}} f_0'\Big(x-\sum_{i=0}^n t_ih_i\Big)(x)\prod_{i=0}^n\phi_i(t_i)dt_0\cdots dt_n$$
$$\geqq 12.$$

Similarly, $f'(-x)(-x)\geqq 12$. Thus, by the implicit function theorem, we have

$$q'(x)=-\left[f'\left(\frac{x}{q(x)}\right)\left(\frac{x}{q(x)}\right)\right]^{-1}f'\left(\frac{x}{q(x)}\right).$$

When restricted to finite dimensional subspaces, f'' exists in the Frechet sense and is continuous. Thus, it follows that q has a second Gateaux derivative on $X\setminus\{0\}$ and q' is locally Lipschitzian on the sphere.

We note that "locally" can be dropped in the hypothesis and conclusion of Theorem 3.1.

In the sequel we will need the following.

3.2 THEOREM *If X is* US (*resp.*, LS), *then X admits a differentiable norm whose differential is uniformly continuous* (*resp., Lipschitzian*) *on the unit sphere.*

PROOF. Assume that X is LS. (The proof for the case when X is US is similar.) Let $\phi: X\to\mathbf{R}$ be differentiable and symmetric, with ϕ' Lipschitzian, $\phi\leqq 0$, $\inf\phi=-1=\phi(0)$ and $\operatorname{supp}\phi\subseteq B(0,\frac12)$. Then $2\|x\|-2\leqq\phi(x)$. Define, for $t>0$,

$$\omega_\phi(t)=\sup\left\{\frac{\phi(x+h)+\phi(x-h)-2\phi(x)}{\|h\|}: x\in X, \|h\|<t\right\}.$$

Let $\psi: B(0,1)\to\mathbf{R}$ be defined by

$$\psi(x)=\inf\left\{\sum_{i=1}^n\alpha_i\phi(x_i):\sum\alpha_ix_i=x,\alpha_i\geqq 0,\sum\alpha_i=1,x_i\in B(0,1),n\in\mathbf{N}\right\}.$$

Let $G=\{x\in B(0,1):\psi(x)<-\frac12\}$.

We claim that ψ has a Lipschitz differential on G. To see this, let $x \in G$, $h \in X$, $\|h\| < \frac{1}{4}$, $h \neq 0$ and $0 < \varepsilon < -\frac{1}{2} - \psi(x)$ be given. Let $x_i \in B(0,1)$, $\alpha_i \geqq 0$, $\sum_{i=1}^{n} \alpha_i = 1$ be such that $\sum_{i=1}^{n} \alpha_i \phi(x_i) < \psi(x) + \varepsilon \|h\| < -\frac{1}{2}$ and assume that $\phi(x_i) < 0$, $i = 1, 2, \cdots, k$ and $\phi(x_{k+1}) = \cdots = \phi(x_n) = 0$. Then

$$-\frac{1}{2} \geqq \sum_{i=1}^{k} \alpha_i \phi(x_i) \geqq -\sum_{i=1}^{k} \alpha_i,$$

so $1 \geqq \alpha = \sum_{i=1}^{k} \alpha_i \geqq \frac{1}{2}$ and

$$\left\| x_i + \frac{1}{\alpha} h \right\| \leqq \|x_i\| + 2\|h\| \leqq 1 \qquad \text{for } i = 1, \cdots, k.$$

Thus

$$\begin{aligned}
&\psi(x+h) + \psi(x-h) - 2\psi(x) \\
&\leqq \sum_{i=1}^{k} \alpha_i \phi\left(x_i + \frac{1}{\alpha} h\right) + \sum_{i=1}^{k} \alpha_i \phi\left(x_i - \frac{1}{\alpha} h\right) - 2\sum_{i=1}^{n} \alpha_i \phi(x_i) + 2\varepsilon \|h\| \\
&= \sum_{i=1}^{k} \alpha_i \left[\phi\left(x_i + \frac{1}{\alpha} h\right) + \phi\left(x_i - \frac{1}{\alpha} h\right) - 2\phi(x_i) \right] + 2\varepsilon \|h\| \\
&\leqq \alpha \omega_\phi\left(\frac{\|h\|}{\alpha}\right) \cdot \frac{\|h\|}{\alpha} + 2\varepsilon \|h\|.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary,

$$\psi(x+h) + \psi(x-h) - 2\psi(x) < 2\omega_\phi(2\|h\|) \cdot \|h\| \leqq C\|h\|^2.$$

This implies the existence of ψ'.

Now, to substantiate our claim, let $x, y \in G$, $0 < \|x - y\| < \frac{1}{8}$ be given. Then for $h \in X$, $\|h\| = \|x - y\|$, $x + h \in B(0,1)$, $y - (x + h - y) \in B(0,1)$ and, by convexity,

$$\begin{aligned}
(\psi'(x) - \psi'(y))(h) &\leqq \psi(x+h) - \psi(x) - \psi'(y)(h) \\
&= \psi(x+h) - \psi(y) - \psi'(y)(x+h-y) + \psi(y) - \psi(x) + \psi'(y)(x-y) \\
&\leqq \psi(x+h) - \psi(y) - \psi'(y)(x+h-y) \\
&\leqq \psi(y + (x+h-y)) + \psi(y - (x+h-y)) - 2\psi(y) \\
&\leqq 2\omega_\phi(2\|x+h-y\|)\|x+h-y\| \leqq 4\omega_\phi(4\|x-y\|)\|x-y\|.
\end{aligned}$$

By taking the supremum over all $h \in X$, $\|h\| = \|x - y\|$, we obtain that, for $x, y \in G$, $0 < \|x - y\| < \frac{1}{8}$,

$$\|\psi'(x) - \psi'(y)\| \leqq 4\omega_\phi(4\|x - y\|) < C\|x - y\|.$$

Let $Q = \{x \in B(0,1) : \psi(x) \leqq -\frac{3}{4}\}$ and let q be the Minkowski functional of Q. Then $q(x) + q(-x)$ is an equivalent norm on X and, using the properties of ψ and that for $q(x) = 1$,

$$\psi'(x)(x) \geqq \psi(x) - \psi(0) = \frac{1}{4},$$

we can complete the proof as in Theorem 3.1.

3.3 THEOREM. *Either of the following conditions* (i) *or* (ii) *implies that* X *is super-reflexive*:

(i) X *is* LUS *and no subspace of* X *is isomorphic to* c_0.

(ii) *The norm on* X *has a locally uniformly continuous differential on* $X \setminus \{0\}$ *and the unit ball of* X *has at least one strongly exposed point.*

PROOF. (i) Every non-super-reflexive space contains a non-super-reflexive separable subspace, so it is sufficient to consider separable X. Now each separable X admits an LUR norm, so the result follows from Proposition 2.1 and Theorem 3.2.

(ii) Assume x_0 is a strongly exposed point of $B(0,1)$ and let $\|\cdot\|'$ be uniformly continuous on $B(x_0, \varepsilon)$. Let $H = \{h \in X : \|x_0\|'(h) = 0\}$. Since x_0 is strongly exposed, there is $\delta > 0$ such that for $h \in H$, $\|h\| > \varepsilon$, we have $\|x + h\| \geqq 1 + \delta$. For $h \in H$, let $\phi(h) = \|x_0 + h\| + \|x_0 - h\| - 2$. Set $Q = \{h \in H : \phi(h) \leqq \delta/2\}$ and let q be the Minkowski functional on $B(x_0, \varepsilon)$. As in Theorems 3.1 and 3.2, we can see, by the implicit function theorem, that q is an equivalent norm on H with uniformly continuous differential on its unit sphere. Thus, there is such a norm on X, so X is super-reflexive.

Analogously, one obtains a Lipschitz version of Theorem 3.3 which we state without proof.

3.4 THEOREM. *Either of the following conditions* (i) *or* (ii) *implies that* X *admits an equivalent norm with a Lipschitz differential on its unit sphere*:

(i) X *is separable,* LLS *and no subspace of* X *is isomorphic to* c_0.

(ii) *The norm on* X *has a locally Lipschitz differential on* $X \setminus \{0\}$ *and the unit ball of* X *has at least one exposed point.*

Condition (ii) in the above theorems should be compared with [3] where a similar result is proved for uniform convexity.

Some of the above results can be summarized in the following.

3.5 THEOREM. *If a separable Banach space* X *is* LLS *and no subspace of* X *is* *isomorphic to* c_0, *then* X *admits a twice Gateaux differentiable norm the first Fréchet differential of which is Lipschitzian on the unit sphere.*

3.6 COROLLARY. *If* X *is* LLS *and no subspace of* X *is isomorphic to* c_0, *then* X *is of type 2.*

PROOF. If X is not of type 2 it contains a separable subspace not of type 2. Thus, it suffices to use Proposition 2.1 and Theorem 3.2 and the result of Figiel and Pisier ([8]) stating that X is of type 2 provided the differential of a norm on X is Lipschitzian on its unit sphere. We take the liberty of giving a shorter proof of their result.

Suppose that $\|\cdot\|'$ is Lipschitzian on the unit sphere, that is, by the uniform analogue of Lemma 2.4, there is $C>0$ such that for each $x,\ y \in X$ we have

$$\|x+y\|^2+\|x-y\|^2 \leqq 2\|x\|^2+2C\|y\|^2.$$

We claim that, for each n and each $h_1,\ h_2,\cdots,h_n \in X$,

$$\sum_{\varepsilon_i=\pm1}\left\|\sum_{i=1}^{n}\varepsilon_i h_i\right\|^2 \leqq 2^n\|h_1\|^2+2^nC\sum_{i=2}^{n}\|h_i\|^2.$$

Indeed, suppose that the estimate is true for numbers up to n. Then

$$\begin{aligned}
\sum_{\varepsilon_i=\pm1}\left\|\sum_{i=1}^{n+1}\varepsilon_i h_i\right\|^2 &= \sum_{\varepsilon_i=\pm1}\left(\left\|\sum_{i=1}^{n}\varepsilon_i h_i+h_{n+1}\right\|^2+\left\|\sum_{i=1}^{n}\varepsilon_i h_i-h_{n+1}\right\|^2\right)\\
&\leqq \sum_{\varepsilon_i=\pm1}\left(2\left\|\sum_{i=1}^{n}\varepsilon_i h_i\right\|^2+2C\|h_{n+1}\|^2\right)\\
&=2\sum_{\varepsilon_i=\pm1}\left\|\sum_{i=1}^{n}\varepsilon_i h_i\right\|^2+2^{n+1}C\|h_{n+1}\|^2\\
&\leqq 2\cdot 2^n\|h_1\|^2+2^{n+1}C\sum_{i=2}^{n}\|h_i\|^2+2^{n+1}C\|h_{n+1}\|^2\\
&=2^{n+1}\|h_1\|^2+2^{n+1}C\sum_{i=2}^{n+1}\|h_i\|^2.
\end{aligned}$$

3.7 COROLLARY. *If* X *and* X^* *are both* LLS, *then* X *is isomorphic to a Hilbert space.*

PROOF. By a result of Bessaga and Pelczynski ([1], see [17], p. 103), if X^* contains a subspace isomorphic to c_0, then X contains a complemented subspace isomorphic to l_1. Since l_1 is not LLS ([15], [2]), we see by Corollary 3.5 that both X and X^* are of type 2. Our result now follows from Kwapien's characterization of spaces isomorphic to Hilbert spaces, namely, spaces X such that X and X^* are both of type 2.

REMARKS. (1) Corollary 3.6 generalizes Meshkov's result [21] that if X, X^* both admit C^2-bump functions then X is isomorphic to Hilbert space.

(2) By Proposition 2.1 and Theorem 3.2, there is no LUR norm on c_0 with locally uniformly continuous differential on the unit sphere. Thus, in general, no averaging result, like Asplund's, which holds for C^1-norms, can be expected for C^2-norms. Therefore the result of [22], which states that there is an LUR norm on c_0 which is a uniform limit of C^∞-norms, cannot be strengthened, in some sense.

(3) If $p > 2$, then $X = (\sum_{n=1}^{\infty} \oplus l_p^n)_2$ is an example of a Banach space whose norm is Lipschitz differentiable ([8], [9]) and which cannot be renormed with a C^2-norm. The latter fact can be proved using the methods used in [20]. Thus, Theorem 3.1 cannot be strengthened to show the existence of a C^2-norm.

(4) In [12] James constructed a nonreflexive space X of type 2. By Theorem 3.3, this space is not an LUS space.

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